Paper 0: Geometric Foundations of the LMR Grammar

Length–Mass Reduction (LMR) Theory

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Abstract

This paper derives the geometric foundation of the Length–Mass Reduction grammar from first principles. Beginning from four postulates concerning the minimal volumetrically closed solid, we show that the regular spherical tetrahedron is the unique geometry satisfying all constraints simultaneously. The tetrahedral structure admits exactly three distinct complementary face-pairing schedules under full face equivalence, a result that determines the structural multiplicity of the lattice's nodal geometry. Observational access from any generic external direction is bounded to three of four faces, providing a geometric basis for three-dimensional structural space. The geometric basis for the involution operator and the corridor algebra of the hourglass diagram is established as a consequence of the tetrahedral symmetry structure. No physical postulates, dynamical laws, or empirical inputs are introduced. The results establish the geometric foundation upon which the structural grammar of Papers I–V rests.

Contents

1 Introduction

The structural grammar of Length–Mass Reduction theory—the hourglass diagram, its corridor operators, the mirror involution mid_1, and the quadrant routing rules—was introduced in Paper I as a fixed codex. That codex was not derived; it was defined and then locked. Subsequent papers populated the grammar with structural configurations and demonstrated its consequences, but the grammar itself was axiomatic.

This paper asks the prior question: why this grammar and not some other? Is the hourglass a convenient organizational choice, or is it the unique diagrammatic representation forced by the geometry of the lattice's minimal structural unit?

The approach follows the tradition of Euclid's *Elements*: begin with definitions of primitive objects, state a small number of irreducible postulates, and derive all subsequent structure as propositions. The goal is to show that the hourglass grammar—including its four quadrant positions, its corridor operators, its mirror involution, and its routing rules—emerges as the minimal faithful representation of a single geometric object: the regular spherical tetrahedron.

No physical mechanism, empirical measurement, or dynamical law is invoked at any point. The paper operates entirely within classical geometry and the theory of finite symmetry groups. Its results are mathematical, not physical. The identification of these geometric results with the structural grammar of LMR is stated in the final section as a representation assignment, not as a derivation of physics from geometry.

2 Scope and Relationship to the Structural Arc

This paper is logically prior to Paper I but was written after the completion of the structural arc (Papers I–V). It does not modify, extend, or replace any result established in those papers. Its purpose is to provide geometric justification for the grammatical structures that Papers I–V assumed.

The structural arc proceeds from grammar to population to consequence: Paper I defines the grammar, Paper II establishes persistence, Paper III classifies configurations, Paper IV develops electromagnetic projection, and Paper V introduces gravitational normalization. At no point does the structural arc derive its own grammar. The present paper fills that gap.

Results established here may be read independently of the structural arc. A reader familiar with classical geometry and finite group theory but unfamiliar with LMR will find the propositions self-contained. The final section provides the explicit mapping between geometric results and LMR grammar for readers who wish to connect the two.

3 Definitions

Definition 1 (Sphere)**.** A sphere is the set of all points equidistant from a center in three-dimensional space. The total solid angle subtended by a sphere from its center is 4π steradians.

Definition 2 (Face)**.** A face is a simply connected region on the surface of a sphere. A face subtends a definite solid angle from the center. A face is the primitive bounding element of the geometry.

Definition 3 (Edge)**.** An edge is the boundary arc shared between two adjacent faces on the surface of a sphere. An edge is a segment of a great circle.

Definition 4 (Vertex)**.** A vertex is a point where three or more edges meet on the surface of a sphere.

Definition 5 (Spherical solid)**.** A spherical solid is a partition of a sphere into a finite number of faces, bounded by edges, meeting at vertices. The partition is exhaustive (every point on the sphere belongs to exactly one face, edge, or vertex) and the faces collectively tile the sphere without gap or overlap.

Definition 6 (Volumetric closure)**.** A spherical solid achieves volumetric closure when its faces collectively tile the full 4π steradians of the sphere. Volumetric closure is the minimal condition for a complete interior–exterior distinction.

Definition 7 (Face equivalence)**.** Two faces of a spherical solid are equivalent if there exists a symmetry operation of the solid that maps one face to the other. A spherical solid has full face equivalence if all of its faces are mutually equivalent under the symmetry group.

Definition 8 (Opposition ranking)**.** For a given external direction $\mathbf{d}$ and a spherical solid, the opposition ranking of the faces is the ordering by the angle between $\mathbf{d}$ and each face's outward normal. The face whose normal is most nearly antiparallel to $\mathbf{d}$ is the most-opposed face. The remaining faces constitute the complementary partition.

Definition 9 (Face-pairing schedule)**.** A face-pairing schedule of a spherical solid is a partition of its faces into complementary pairs. Two face-pairing schedules are distinct if they assign different pairs. Two distinct schedules are symmetry-equivalent if a symmetry operation of the solid maps one to the other; they are symmetry-inequivalent otherwise.

Definition 10 (Involution)**.** An involution is an operation σ satisfying $\sigma^2 = \text{id}$. Applied twice, it returns to the identity.

Definition 11 (Face-edge unit)**.** A face-edge unit is the structural primitive consisting of one face together with the edges in which it participates. The face carries a solid angle (curvature content). The edges carry relational content connecting the face to its neighbors. Neither component is structurally complete without the other: a face without edges has no connection to adjacent structure, and an edge without faces has no bounding region to relate.

4 Postulates

4.1 Part A: Geometric Postulates

Postulate 1 (Closure)**.** An admissible partition of the sphere is one whose faces exhaust the full 4π steradians of the spherical boundary without gap or overlap.

Postulate 2 (Minimality)**.** Among admissible closed partitions, the minimal partition has the fewest faces consistent with closure.

Postulate 3 (Face equivalence)**.** The minimal admissible partition is face-transitive: every face is equivalent to every other face under the symmetry of the partition.

Postulate 4 (Connected geodesic faces)**.** Each face of the admissible partition is a simply connected spherical polygon bounded by at least three arcs of great circles.

4.2 Part B: Structural Enrichment Postulates

The following postulates are introduced after the tetrahedral geometry has been derived from Postulates 1–4. They add structural content to the already-established geometry.

Postulate A (Central exclusion)**.** The enclosed interior of the minimal partition is not an admissible routing domain. Structural continuation occurs only through the boundary organization of faces, edges, and vertices.

Postulate B (Intrinsic edge orientation)**.** Each edge of the partition carries an intrinsic orientation—a directional asymmetry distinguishing traversal from one adjacent face to the other. This orientation is not imposed externally; it is a primitive property of the edge.

5 The Minimal Closed Solid

Proposition 1. *The minimum number of faces for a partition of the sphere satisfying Postulates 1–4 is four.*

Proof. $N = 1$**.** A single face has no boundary edges and cannot be bounded by at least three great circle arcs. Excluded by Postulate 4.

$N = 2$**.** Two faces sharing the sphere must meet along a common boundary. A single great circle divides the sphere into two hemispheres, each bounded by one arc—violating the three-arc minimum of Postulate 4. To achieve at least three bounding arcs per face with only two faces, each face requires at least three edges and hence at least three vertices. But with only two faces, every edge separates the same pair. A spherical triangle and its complement cannot be equivalent under any symmetry—one subtends less than 2π steradians, the other more—violating face equivalence (Postulate 3). Excluded.

$N = 3$**.** By Postulate 3, all three faces are equivalent, so each has the same number k of boundary edges. By Postulate 4, $k \geq 3$. Each edge borders exactly two faces, so the total face-edge incidence count satisfies

$$3k = 2E.$$

The left side must be even, so k must be even. Therefore $k \neq 3$, and the faces cannot be spherical triangles. The minimum even value is $k = 4$, giving $E = 6$.

However, with only three faces, each face is adjacent to both others. No non-adjacent face pairs exist. The boundary of each face must alternate between the only two neighboring face labels, which forces an even number of boundary arcs—consistent with k even, but imposing a rigid constraint on boundary structure.

The minimal face-transitive geodesic tripartition is the three-lune partition.

$N = 4$ **is achievable.** Four spherical triangles of equal solid angle π, arranged as the regular spherical tetrahedron, satisfy all four postulates: they exhaust 4π (Postulate 1), four is the face count (Postulate 2), all faces are equivalent under the tetrahedral symmetry group T_d (Postulate 3), and each face is a spherical triangle bounded by exactly three great circle arcs (Postulate 4).

Since $N = 1, 2, 3$ are excluded and $N = 4$ is achievable, the minimum is four. ☐

Proposition 2. *The unique partition of the sphere into four faces satisfying Postulates 1–4 is the regular spherical tetrahedron.*

Proof. By Proposition 1, the minimal admissible face count is $N = 4$.

Step 1: Edge count. Each face has k boundary edges, with $k \geq 3$ (Postulate 4). By face equivalence (Postulate 3), all four faces share the same k. Each edge borders exactly two faces, so

$$4k = 2E,$$

giving $E = 2k$.

Step 2: Minimal boundary complexity. By Postulate 4, $k \geq 3$. Since $F = 4$ is already the minimal admissible face count by Proposition 1 and Postulate 2, the admissible four-face partition must realize the minimal possible boundary complexity per face. Hence $k = 3$. Each face is a spherical triangle, and $E = 6$.

Step 3: Vertex count. By Euler's formula for spherical polyhedra ($V - E + F = 2$), with $F = 4$ and $E = 6$:

$$V = 2 - F + E = 2 - 4 + 6 = 4.$$

Step 4: Vertex valence. Each edge terminates at two vertices, giving $2E = 12$ total vertex-edge incidences distributed over $V = 4$ vertices, for an average valence of 3. Since all faces are equivalent and triangular, every face contributes the same local incidence pattern. Any deviation from constant valence would force unequal local face neighborhoods, contradicting face equivalence in the four-face triangular case. Hence each vertex is trivalent.

Step 5: Uniqueness. We have established a four-face spherical partition with six edges, four vertices, triangular faces, and trivalent vertices. This is the tetrahedral combinatorial type. By Postulate 3, the symmetry group acts transitively on faces, so all faces have identical incidence structure. Corresponding edges and angles must agree across faces, making the four spherical triangles congruent. The unique face-transitive realization of this combinatorial type on the sphere is the regular spherical tetrahedron, with symmetry group T_d. □

Proposition 3. *Each face of the regular spherical tetrahedron subtends exactly π steradians from the center of the sphere.*

Proof. By Proposition 2, the regular spherical tetrahedron partitions the sphere into four congruent faces. By Postulate 1, the four faces exhaust the full 4π steradians. By congruence, each face subtends the same solid angle. Therefore each face subtends

$$\frac{4\pi}{4} = \pi \text{ steradians.}$$

Independently, the solid angle of a spherical triangle equals its spherical excess: the sum of its interior angles minus π. For the regular spherical tetrahedron, the side length a of each spherical face is the central angle between adjacent tetrahedral vertices, so $\cos a = -1/3$. By the spherical law of cosines for equilateral spherical triangles, each interior angle A satisfies

$$\cos A = \frac{\cos a}{1 + \cos a}.$$

Substituting $\cos a = -1/3$:

$$\cos A = \frac{-1/3}{1 - 1/3} = \frac{-1/3}{2/3} = -\frac{1}{2},$$

so $A = 2\pi/3$. The spherical excess is

$$3A - \pi = 3\left(\frac{2\pi}{3}\right) - \pi = 2\pi - \pi = \pi.$$

□

Proposition 4. *The four faces of the regular spherical tetrahedron admit exactly three distinct complementary face-pairing schedules. Each schedule corresponds to one pair of opposite edges of the tetrahedron.*

Proof. **Step 1: Combinatorial count.** Label the four faces F_1, F_2, F_3, F_4. A complementary face-pairing schedule is a partition of the four faces into two disjoint pairs. The number of ways to choose the first pair is $\binom{4}{2} = 6$. Each choice determines its complement, so each schedule is counted twice. Hence the number of distinct schedules is

$$\frac{\binom{4}{2}}{2} = 3.$$

Explicitly:

- Schedule I: $(F_1F_2)(F_3F_4)$
- Schedule II: $(F_1F_3)(F_2F_4)$
- Schedule III: $(F_1F_4)(F_2F_3)$

Step 2: Geometric realization. In a tetrahedron, each edge is shared by exactly two faces. Thus each pair (F_iF_j) identifies the edge common to those two faces. The three pairing schedules correspond to the three ways of selecting a pair of opposite edges—edges sharing no vertex.

Each pair of opposite edges determines one of the three 2-fold rotational symmetry axes of the tetrahedron. The three complementary face-pairing schedules are therefore realized geometrically as the three opposite-edge pairings of the tetrahedron.

Step 3: Symmetry relation. The tetrahedral symmetry group permutes these three schedules transitively. They are distinct as schedules but symmetry-equivalent as realizations of the regular tetrahedron.

Since the combinatorial count is exactly three and each schedule is realized geometrically by one opposite-edge pair, the result follows. □

Corollary 1. *The regular spherical tetrahedron has structural multiplicity three at the level of distinct complementary pairing schedules, even though these schedules are symmetry-equivalent in the ideal tetrahedron. External structural context may break this symmetry and render the three schedules operationally distinguishable.*

Proposition 5. *For any generic external direction, the four face normals of the regular spherical tetrahedron determine a unique most-opposed face (Definition 8). The remaining three faces form the complementary three-face partition. This $3 + 1$ decomposition holds for all directions except a measure-zero set of tie directions.*

Proof. **Step 1: Opposition ranking.** For any external direction $\mathbf{d}$, the angle between $\mathbf{d}$ and each of the four outward face normals defines an opposition ranking (Definition 8). The face whose normal is most nearly antiparallel to $\mathbf{d}$ is the most-opposed face.

Step 2: Uniqueness for generic directions. The four outward normals of the regular spherical tetrahedron are tetrahedrally distributed, with pairwise angular separation $\arccos(-1/3) \approx 109.47^\circ$. For a generic direction $\mathbf{d}$, the four face-normal opposition angles are distinct, and exactly one face achieves the maximum opposition angle. This face is uniquely most-opposed.

Step 3: Stable three-face partition. The remaining three faces—those not most-opposed—constitute the complementary partition. This partition is stable in the sense that small perturbations of $\mathbf{d}$ do not change which face is most-opposed, except at the measure-zero boundary between opposition regions.

Step 4: Measure-zero ambiguity. Tie directions—where two or more faces share the maximum opposition angle—form a set of measure zero on the sphere. These correspond to directions exactly equidistant (in angular terms) from two or more face normals. For all other directions, the $3 + 1$ decomposition is unique and stable. ☐

Corollary 2. *The regular spherical tetrahedron generically admits a stable* $3+1$ *opposition partition: one uniquely most-opposed face and three complementary faces. This partition is determined by tetrahedral geometry and is independent of external labeling.*

6 Structural Enrichment

The regular spherical tetrahedron has been derived from the geometric postulates (Postulates 1–4). We now introduce two structural postulates that enrich the bare geometry with properties necessary for the representation of physical structure. These postulates do not alter the geometric results of Propositions 1–5. They add structural content to the already-established geometry.

Proposition 6. *Under Structural Postulate A, the regular spherical tetrahedron is centrally exclusive: its enclosed interior is not an admissible routing domain, and all structural continuation is boundary-mediated.*

Proof. By Propositions 1–5, the regular spherical tetrahedron is a closed boundary structure with four faces, six edges, and four vertices. Structural Postulate A states that the enclosed interior of this boundary is excluded from admissible routing. Therefore any structural continuation between parts of the tetrahedral node must occur through the boundary organization itself.

In the tetrahedron, the boundary organization consists of: faces, which present the local surface domains; edges, which mediate pairwise face relations; and vertices, where edge relations meet.

Since the interior is excluded as a routing medium, no admissible continuation may pass directly through the enclosed center. All continuation is boundary-mediated. The regular spherical tetrahedron is centrally exclusive in the required structural sense. ☐

Proposition 7. *Under Structural Postulate B, the six edges of the centrally exclusive regular spherical tetrahedron carry intrinsic orientation. These six oriented edges partition into three complementary pairs of opposite edges, each supplying the geometric support for one structural corridor and corresponding to one of the three distinct face-pairing schedules of Proposition 4.*

Proof. **Step 1: Edge orientation.** By Structural Postulate B, each of the six edges carries an intrinsic orientation—a directional asymmetry distinguishing traversal from one adjacent face to the other. This orientation is a primitive property of the edge, not derived from external assignment.

Step 2: Opposite edges. Two edges of a tetrahedron are opposite if they share no vertex. The regular tetrahedron has six edges and four vertices, with each vertex incident to three edges. The three pairs of opposite edges are:

- Pair I: edge (F_1F_2) and edge (F_3F_4)
- Pair II: edge (F_1F_3) and edge (F_2F_4)
- Pair III: edge (F_1F_4) and edge (F_2F_3)

where (F_iF_j) denotes the edge shared by faces F_i and F_j. Each pair of opposite edges determines one of the three 2-fold rotational symmetry axes.

Step 3: Corridor supports. Each pair of opposite oriented edges, equipped with intrinsic orientation, supplies the geometric support for one structural corridor across the boundary network. The two edges in an opposite pair are maximally separated on the tetrahedral boundary—they share no vertex and no face—yet they are complementary in the sense that together they account for all four faces (each face is incident to exactly one edge of the pair).

Since the tetrahedron admits exactly three pairs of opposite edges, and each pair supplies the geometric support for one structural corridor, the centrally exclusive regular spherical tetrahedron admits exactly three structural corridor supports. □

Corollary 3. *Each face of the regular spherical tetrahedron is incident to three edges and is represented exactly once in each of the three opposite-edge pairs. This motivates the face-edge unit as the natural structural primitive: a π-steradian face patch carrying curvature content and participating in the three oriented edge relations that carry directional content.*

7 The Involution

With central exclusion fixing the interior as non-routing and opposite-edge orientation supplying three boundary-mediated corridor supports, the remaining geometric task is to determine how visible and hidden face structure are related under boundary exchange.

Proposition 8. *The regular spherical tetrahedron admits three natural involutions, each determined by one pair of opposite edges. Each involution exchanges the two complementary face-pairs associated with that opposite-edge pair, is self-inverse, and acts through the boundary structure rather than through the excluded interior.*

Proof. **Step 1: Three complementary face-pairing schedules.** By Proposition 4, the four faces admit exactly three distinct complementary face-pairing schedules: $(F_1F_2)(F_3F_4)$, $(F_1F_3)(F_2F_4)$, and $(F_1F_4)(F_2F_3)$.

Step 2: Opposite-edge realization. By Proposition 7, each schedule corresponds to one pair of opposite edges of the tetrahedron. Each pair of opposite edges defines a unique axis passing through the midpoints of both edges.

Step 3: Involution by rotation. A 180° rotation about the axis defined by a pair of opposite edges exchanges the two faces in each complementary pair while preserving the opposite-edge axis itself. Denote this rotation σ. Under σ: the two faces paired in one complementary pair exchange positions; the two faces in the other pair also exchange positions; and the two opposite edges defining the axis are preserved.

Step 4: Self-inverse. Applying σ twice returns every face to its original position: $\sigma^2 = \text{id}$. The operation is an involution.

Step 5: Boundary-mediated. By Proposition 6 (central exclusion), the enclosed interior is not an admissible routing domain. The rotation σ acts on the boundary structure—faces, edges, and vertices—without invoking interior passage. The exchange is entirely boundary-mediated.

Since there are exactly three opposite-edge pairs (Proposition 4), there are exactly three such involutions. They are symmetry-equivalent under T_d but distinct as operations. □

Corollary 4. *Each involution is a boundary-mediated self-inverse operation that exchanges complementary face-pairs while preserving the opposite-edge axis that determines it.*

Remark 7.1 (Representational reading). The involution acts most naturally as a face exchange supported by an invariant opposite-edge scaffold. In this reading, the oriented edges remain the structural support of the exchange while the face patches are the exchanged curvature-bearing elements. This motivates—but does not yet formally establish—a representation in which edge structure is the persistent scaffold and face structure is the exchanged content.

Remark 7.2 (Relation to observational access). For a generic external direction, Proposition 5 establishes a three-accessible, one-hidden face partition. The involutions of Proposition 8 provide a boundary-mediated mechanism by which hidden structure may become accessible under a change of structural perspective. The specific accessible/hidden assignment is direction-dependent, but the involutive exchange structure is fixed by the tetrahedral boundary. The precise relationship between viewpoint-dependent accessibility and schedule-dependent involution is deferred to the representation layer.

Remark 7.3 (Boundary-unit lens). The face-edge unit introduced in Definition 11 admits two structural readings corresponding to the two sides of the spherical boundary:

- The **M'-face** (interior-facing) records curvature and closure on the enclosed side of the spherical boundary. Each face patch subtends π steradians of the interior-facing surface.
- The **λ-face** (exterior-facing) records corridor and orientational content on the outward side of the same boundary unit. The edge relations that connect adjacent face patches are read from this exterior-facing side.

These are two structural readings of the same boundary unit, not two detached objects occupying separate regions. The involution of Proposition 8 provides the geometric basis for representing the same boundary unit under interior-facing (M') and exterior-facing (λ) readings. The precise identification of involutive exchange with the M'/λ correspondence is deferred to the representation layer.

8 The Hourglass as Minimal Representation

Proposition 9. *The tetrahedral boundary unit with two structural readings and involutive exchange requires a representation with at least four distinct positions.*

Proof. The regular spherical tetrahedron has four face-edge units (Propositions 1–2, Definition 11). Each face-edge unit admits two structural readings: an interior-facing reading and an exterior-facing reading (Remark 7.3).

A faithful representation must distinguish at least four face-edge units while preserving the involutive exchange structure (Proposition 8). Fewer than four positions would require identifying two or more face-edge units, collapsing the structural distinctions established by the tetrahedral geometry.

Furthermore, the involution σ acts on these four units by exchanging complementary pairs (Proposition 8). A representation with only two positions could encode one complementary pair but not the full four-unit structure. A representation with three positions would break the pair symmetry of the involution, which requires an even partition.

Four positions is therefore the minimum. ☐

Proposition 10. *The minimal planar representation of the tetrahedral exchange structure is a crossed four-position diagram with an excluded center.*

Proof. **Step 1: Four positions required.** By Proposition 9, a faithful representation requires at least four distinct positions.

Step 2: Crossed pair exchange. The involution of Proposition 8 requires a self-inverse exchange between complementary position pairs. In a planar four-position diagram, a natural minimal faithful encoding of such an exchange is a crossed pairing structure. The crossing visually represents the self-inverse exchange relation without collapsing the four distinct positions. Applying the crossing twice returns all positions to their original assignments, consistent with $\sigma^2 = \text{id}$.

Step 3: Excluded center. The point where the crossing occurs—the center of the diagram—is the natural planar image of the excluded interior of the centrally exclusive tetrahedron (Proposition 6). The center carries no structural content and serves only as the organizing point of the exchange geometry.

Step 4: Representational readiness. The crossed four-position diagram admits natural horizontal and vertical relations between its positions, making it suitable for later assignment of invariant and exchanged structural roles. These assignments are not determined by the present proposition and are deferred to the representation layer.

Step 5: Minimality. No planar representation with fewer than four positions preserves all of the following simultaneously: four distinct structural units, complementary pair structure, self-inverse crossed exchange, and an excluded center. The crossed four-position diagram with excluded center is therefore the minimal faithful planar representation. ☐

Corollary 5. *The hourglass diagram is not an independently chosen organizational tool. It is the minimal planar representation of the tetrahedral boundary exchange algebra derived in Propositions 1–8.*

Corollary 6. *The hourglass diagram is representation-ready: it distinguishes four positions, supports self-inverse pair exchange, and preserves an excluded center. This makes it suitable for later assignment of invariant corridor structure and exchanged boundary readings.*

9 Bridge to the Structural Grammar

The preceding sections derived the regular spherical tetrahedron as the unique minimal geometry satisfying the geometric postulates, and the hourglass as its minimal planar representation. This section assigns structural roles to the hourglass positions, drawing on the tetrahedral geometry and the boundary-lens distinction of Remark 7.3. These assignments constitute the representation layer: they map geometric results to the grammatical positions used in the LMR structural arc (Papers I–V).

Assignment 9.1 (Vertical axis: boundary exchange)**.** The vertical axis of the hourglass represents the exchange between interior-facing and exterior-facing readings of the boundary unit. Upper positions record one reading; lower positions record the complementary reading under the involution. This vertical exchange is a geometric basis for the operator mid_1 defined in Paper I.

Assignment 9.2 (Horizontal axis: invariant corridor scaffold)**.** The horizontal axis of the hourglass represents the structural relation that the involution preserves—the edge scaffold that supports the face exchange. Positions sharing a horizontal level belong to the same complementary pair and are not exchanged by the involution. This invariant horizontal relation is a geometric basis for the c-corridor defined in Paper I.

Assignment 9.3 (Diagonal crossing: the involution σ)**.** The diagonal crossing of the hourglass represents the involution σ of Proposition 8. It exchanges complementary face-pairs across the boundary. In the LMR grammar, this crossing corresponds to the mid_1 involution acting on occupied positions.

Assignment 9.4 (Center: excluded interior as unity)**.** In the LMR representation layer, the center of the hourglass is assigned the value unity, reflecting its role as the identity element under the involution: $\sigma(1) = 1$. The center carries no structural content (Proposition 10, Step 3).

Assignment 9.5 (Position labels)**.** Under the boundary-lens distinction:

- **Upper-left (UL):** Interior-facing, primary. Assigned to M' (inverse-length, curvature/closure content).
- **Upper-right (UR):** Interior-facing, secondary. Assigned to f (frequency, the corridor-transformed partner of M' under the horizontal relation: $M' \times c = f$).
- **Lower-right (LR):** Exterior-facing, primary. Assigned to λ (length, corridor/orientational content). Related to UL by the involution: $M' \leftrightarrow \lambda$ under mid_1.
- **Lower-left (LL):** Exterior-facing, secondary. Assigned to t (period, the corridor-transformed partner of λ under the horizontal relation: $t \times c = \lambda$). Related to UR by the involution: $f \leftrightarrow t$ under mid_1.

Remark 9.1 (Structural accessibility hypothesis). The generic 3+1 opposition partition of Proposition 5 supports a structural reading in which three independent directions are perspective-accessible from any single viewpoint while one direction remains complementary. Under this reading, the maximum number of simultaneously accessible independent structural directions presented by the tetrahedral node is three. The identification of this structural accessibility with three-dimensional space is a representation-layer hypothesis, not a geometric theorem. It is listed as such in the correspondence table.

Proposition 11. *The hourglass position assignments of Section 9 are consistent with the tetrahedral geometry derived in Propositions 1–8 and reproduce the grammatical structure defined in Paper I.*

Proof. The vertical exchange (Assignment 9.1) maps UL $\leftrightarrow$ LR and UR $\leftrightarrow$ LL, consistent with the self-inverse diagonal crossing derived in Proposition 10 and with the mid_1 involution of Paper I.

The horizontal relation (Assignment 9.2) connects UL to UR and LL to LR by a single invariant operation, consistent with the preserved edge scaffold and with the c-corridor of Paper I.

The center (Assignment 9.4) carries no structural content and acts as the identity, consistent with central exclusion (Proposition 6) and with $\mathrm{mid}_1 = 1$ at the center of the Paper I hourglass.

The four positions (Assignment 9.5) carry the quantities M', f, t, λ, which satisfy:

$$M' \times c = f \qquad \text{(upper horizontal)}, \tag{1}$$

$$t \times c = \lambda \qquad \text{(lower horizontal)}, \tag{2}$$

$$M' = 1/\lambda \qquad \text{(vertical inversion, } \mathrm{mid}_1\text{)}, \tag{3}$$

$$f = 1/t \qquad \text{(vertical inversion, } \mathrm{mid}_1\text{)}. \tag{4}$$

These are exactly the corridor relations defined in Paper I, Section 4.3. The hourglass grammar of Paper I is therefore a faithful minimal representation of the tetrahedral exchange algebra under the assignments of Section 9. □

Corollary 7. *The hourglass grammar of Paper I is not a convention. It is a faithful minimal planar representation of the regular spherical tetrahedron's boundary exchange algebra, equipped with the structural assignments introduced in Section 9 and grounded in the tetrahedral geometry and boundary-lens distinction.*

10 Conclusion

The regular spherical tetrahedron has been established from the geometric postulates as the unique minimal face-transitive geodesic partition of the sphere. Its properties—four congruent faces each subtending exactly π steradians, three distinct complementary face-pairing schedules, three-face observational access from any generic direction, and six edges partitioned into three opposite-edge pairs—follow as propositions from the postulates alone.

Two structural enrichment postulates—central exclusion of the interior and intrinsic edge orientation—give the bare geometry its structural character: a boundary-organized node whose interior is not a routing domain and whose edges carry directional content.

The involution, derived from the opposite-edge rotation structure, provides a self-inverse boundary exchange between complementary face-pairs. The hourglass diagram emerges as the minimal faithful planar representation of this exchange algebra: four positions, a crossed pair-exchange, and an excluded center.

Under the structural assignments of Section 9, the hourglass positions correspond to the grammatical slots of the LMR codex (Paper I), and the corridor relations of the structural arc (Papers I–V) are reproduced as consequences of the tetrahedral exchange algebra.

The correspondences listed below are not all of the same logical type. Some are geometric theorems established in this paper; others are structural enrichments or representation-layer assignments introduced to connect the geometry to the LMR codex. Where a correspondence remains interpretive or programmatic, it is labeled accordingly.

Geometric Result	**LMR Grammar Element**	**Status**
4 face-edge units	4 hourglass positions (UL, UR, LL, LR)	Geometric theorem → representation assignment
π steradians per face	π as half-fold curvature (M' face)	Geometric theorem
4π total spherical boundary	Volumetric closure → mid_1	Geometric theorem → representation assignment
3 distinct face-pairing schedules	3 basin facings	Geometric theorem → structural assignment
Generic 3+1 opposition partition	Maximum of 3 perspective-side structural directions	Geometric theorem → structural hypothesis
Central exclusion of interior	Basin closure (interior not a routing domain)	Structural enrichment
Intrinsic edge orientation	Half-fold orientational content (λ face, $\sqrt{c}$)	Structural enrichment
3 opposite-edge pairs	3 structural corridor supports	Structural enrichment → representation assignment
Involution σ	mid_1 (boundary exchange, self-inverse)	Geometric theorem → representation assignment
Excluded center	Unity at mid_1	Representation assignment
Hourglass diagram	LMR codex grammar (Paper I)	Representation assignment

Scope. This paper derives the geometric foundation of the LMR grammar. It does not derive physics. The identification of geometric results with physical structure—particles, forces, interactions—is the work of the structural arc (Papers I–V) and subsequent developments.

The three-face observational access result (Proposition 5) is listed in the correspondence table as a structural hypothesis regarding spatial dimensionality. This identification is geometrically motivated but not formally proven within the present paper.

No dynamical laws, field equations, empirical measurements, or physical postulates appear at any point. The results are mathematical. Their correspondence with the LMR grammar is stated as a representation assignment, not as a derivation of physics from geometry.

This paper belongs to the Length–Mass Reduction (LMR) program. It is logically prior to Paper I and provides the geometric foundation for the structural grammar defined therein. It introduces no physical primitives, dynamical claims, or empirical content.

How to Cite This Paper

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